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Non-linear capillary waves generated by steep gravity waves

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The train of capillary waves which appears on the forward face of a steep gravity wave is discussed by considering the capillary waves as stationary waves on a slowly varying running stream, using the non-linear capillary wave solution of Crapper (1957) and the recent method due to Whitham (1965*a*, *b*). Then energy input and damping are introduced into an energy equation which becomes a non-linear ordinary differential equation for the capillary wave steepness. Numerical solutions for various gravity wavelengths and steepnesses are discussed, and some light is thrown on the problem of the breaking of gravity waves.

1. Introduction

It is well known that a steep gravity wave often carries with it a train of capillary waves near to its crest on its forward face. The formation of these waves has been discussed by Longuet-Higgins (1963). It is suggested in that paper that the capillary waves are generated by the normal stress associated with the effect of surface tension near the crest of the gravity wave, a pressure $-\gamma\kappa$, where γ is the surface tension and κ the curvature of the gravity wave surface. The wavelength and amplitude of the capillary waves are then calculated as a linear perturbation on the non-linear gravity wave. Although this approach gives some useful results it has a major drawback, in that, as the calculated capillary waves are linear, their amplitude is severely restricted. Capillary waves which are large enough to be seen are almost cetainly non-linear.

In the present theory the capillary waves are first of all (§2) regarded as non-linear stationary waves on a variable stream, and the recent method due to Whitham (1965*a*, *b*) is used to construct an equation which gives the wavelength and amplitude at any point in terms of an arbitrary constant, C. This is in a system with no input of energy and no dissipation. We then turn (§3) to the energy equation and allow both input and dissipation, with C becoming a variable. The energy equation becomes a non-linear ordinary differential equation for C which is easily solved numerically when the stream is known. The stream is taken to be an approximate form of a gravity wave, the same one used and discussed by Longuet-Higgins (his §4), which is unfortunately not as accurate as one would like, but probably the best available at the moment. The Whitham theory imposes the restriction that the fluid velocity in the gravity wave must vary by only a small amount over a capillary wavelength, but this has hardly any effect. However, when C is allowed to be a variable this must also be slowly varying in the same sense, so that each capillary wave approximates to one in a uniform wave train, and this condition is in general violated near the crest of the gravity wave.

Results have been calculated for various lengths of gravity wave, and are presented in §4. As expected they show capillary waves forward of the crest of the gravity wave. Under suitable circumstances these capillary waves may 'break' by enclosing air bubbles in their troughs, and it is suggested that this is in fact what is happening when gravity waves are seen to break. The energy dissipated from the gravity wave by these capillaries is found to exceed by a considerable margin that damped out directly by viscosity provided that the gravity wave is sufficiently steep and of fairly short wavelength.

2. Capillary waves on a non-uniform stream

Whitham (1965*a*, *b*) has shown how by use of an 'averaged Lagrangian', $\mathscr{L}$, it is possible to find equations for the variation of wave amplitude, length and frequency in a non-linear dispersive wave system under slowly varying non-uniform conditions. In the simple case of one-dimensional wave propagation with $\mathscr{L}$ expressed as a function of frequency ω and wave-number k only, the appropriate equation for waves propagating over still water is

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial k}\right)=0 \tag{1}$$

(Whitham 1965*b* (26); Lighthill 1965 (3)). In the present problem it is convenient to treat the capillary waves in a frame of reference in which they are stationary. Following Lighthill (1967) we see that if they are stationary on a stream moving with velocity U in the positive x direction, we replace ω in $\mathscr{L}$ by $-Uk$, $\partial/\partial t \equiv 0$, and (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial k}\right)_{U \text{ const}}=0; \tag{2}$$

according to the theory this equation will hold even if U is a function of x provided that U changes only by a small amount over one wavelength. Then we have

$$\partial \mathscr{L}/\partial k = \text{constant} = -C, \tag{3}$$

say.

There remains the problem of finding the appropriate Lagrangian. The problem of non-linear capillary waves on a steady stream, with gravity neglected as a restoring force, was solved by the present author (Crapper 1957) and the appropriate Lagrangian, regarded as the average kinetic energy $\mathscr{T}$ minus the average potential energy $\mathscr{V}$ has already been calculated from that solution by Lighthill (1965 (80)). With ω replaced by $-Uk$ it is

$$\mathscr{L} = 2\gamma-\frac{\rho U^2}{k}-\frac{\gamma^2 k}{\rho U^2}, \tag{4}$$

where γ is the surface tension and ρ the density. Thus (3) gives

$$k = \frac{\rho U^2}{\sqrt{(\gamma^2 - C\rho U^2)}}. \tag{5}$$

In Crapper (1957) a non-dimensional wave-number, there denoted by k but here by $\bar{k}$, is used, where

$$\bar{k} = \frac{k\gamma}{\rho U^2} = \frac{\gamma}{\sqrt{(\gamma^2 - C\rho U^2)}}. \tag{6}$$

Then the wave amplitude a_c (defined as the vertical height between crest and trough) and wavelength λ_c are related by

$$\frac{a_c}{\lambda_c} = \frac{2}{\pi}\sqrt{(\bar{k}^2 - 1)} = \frac{2}{\pi}\sqrt{\left(\frac{C\rho U^2}{\gamma^2 - C\rho U^2}\right)}. \tag{7}$$

There are some restrictions on the constant C, because using the formula for $\mathscr{V}$ calculated by Lighthill (1965) the average kinetic energy is

$$\mathscr{T} = \mathscr{L} + \mathscr{V} = \frac{C\rho U^2}{\sqrt{(\gamma^2 - C\rho U^2)}}. \tag{8}$$

Thus $$0 < C < \gamma^2/\rho U^2. \tag{9}$$

Finally the energy density is

$$\mathscr{E} = \mathscr{L} + 2\mathscr{V} = \frac{C\rho U^2 + 2\gamma^2}{\sqrt{(\gamma^2 - C\rho U^2)}} - 2\gamma. \tag{10}$$

If (5) and (10) are linearized on the assumption that C is small we recover at once two results of Longuet-Higgins: the capillary wave-number is locally that of free capillary waves on a stream of speed $U(\rho U^2/\gamma)$ and the energy density is proportional to U^2.

3. The energy equation

The results of §2 are for a system in which there is no dissipation or input of energy. But for a train of capillary waves the rate of dissipation is high, as is easily observed, and therefore in this section we set out to take account of it.

We consider first the energy equation in the system of §2. Whitham (1965*b* (39), (46)) shows that when $\mathscr{L} = \mathscr{L}(\omega, k)$ the energy density is $\omega\mathscr{L}_\omega - \mathscr{L}$ and that the energy travels at a velocity $-\omega\mathscr{L}_k/(\omega\mathscr{L}_\omega - \mathscr{L})$ relative to still water, the suffixes denoting derivatives. Thus if added to a uniform stream of speed U the convection term in the energy equation is

$$\frac{\partial}{\partial x}\left\{\left(U - \frac{\omega\mathscr{L}_k}{\omega\mathscr{L}_\omega - \mathscr{L}}\right)(\omega\mathscr{L}_\omega - \mathscr{L})\right\}. \tag{11}$$

Now replace $\mathscr{L}(\omega, k)$ by $\mathscr{L}(-Uk, k)$ and (11) becomes

$$\frac{\partial}{\partial x}(Uk\mathscr{L}_k - U\mathscr{L}), \tag{12}$$

where $\mathscr{L}_k$ now means keeping U constant. When U is independent of x and $\partial/\partial t \equiv 0$ this is the only term in the energy equation. However, when U is slowly varying, but still with $\partial/\partial t \equiv 0$, (12) is not zero, but is balanced by an interaction term, R, associated with a tensor which is usually called the 'radiation stress' (Longuet-Higgins & Stewart 1960):

$$\frac{d}{dx}(Uk\mathscr{L}_k - U\mathscr{L}) + R = 0. \tag{13}$$

Now with the results of § 2

$$\mathscr{L} = 2\gamma - \frac{2\gamma^2 - C\rho U^2}{\sqrt{(\gamma^2 - C\rho U^2)}}; \quad \mathscr{L}_k = -C; \tag{14}$$

and therefore from (13) we can calculate R, remembering C = constant. Alternatively we can regard C as a variable and re-write the energy equation (13) as

$$\frac{d}{dx}(Uk\mathscr{L}_k - U\mathscr{L})_{U\,\mathrm{const}} = 0, \tag{15}$$

or, with (14)

$$\frac{-\rho U^3}{\sqrt{(\gamma^2 - C\rho U^2)}}\frac{dC}{dx} = 0. \tag{16}$$

With C constant this is of course an identity, but now we are going to add in energy input and dissipation terms and allow C to be a variable. Provided that C is only slowly varying, in the same way as U, the ideas behind Whitham's theory will remain valid. If we have energy input I per unit length and dissipation D per unit length the energy equation will be

$$\frac{-\rho U^3}{\sqrt{(\gamma^2 - C\rho U^2)}}\frac{dC}{dx} = I - D. \tag{17}$$

It remains to calculate I and D and to solve (17) for C. We can then return to (5), (7) and (10) to give the details of the capillary wave at any point.

To find a form for D we use the result given by Lamb (1932, § 329 (13)), that for an irrotational wave motion, wavelength λ_c,

$$D = \frac{\mu}{\lambda_c}\int_0^{\lambda_c}\frac{\partial q^2}{\partial n}\,ds, \tag{18}$$

where μ is the viscosity, q the velocity, $\partial/\partial n$ the derivative along the outward normal to the surface and ds the length element along the surface. Using the capillary wave solution of Crapper (1957), (18) can be re-written as

$$D = -\frac{\mu kc}{2\pi}\int_0^{2\pi/kc}\left(\frac{\partial q^2}{\partial \psi}\right)_{\psi=0} d\phi, \tag{19}$$

where

$$\left.\begin{aligned} q = c\,e^{\tau}; \quad \frac{\gamma}{\rho c}\frac{\partial \tau}{\partial \psi} &= -\sinh\tau \quad \text{on} \quad \psi = 0; \\ \text{and} \qquad e^{\tau} &= \frac{1 + A^2 + 2A\cos(k\phi/c)}{1 + A^2 - 2A\cos(k\phi/c)} \quad \text{on} \quad \psi = 0. \end{aligned}\right\} \tag{20}$$

The notation here differs from the earlier paper in that dimensional variables have been retained and γ replaces T as surface tension. Using (20) the integral can be evaluated by elementary methods to give

$$D = \frac{4\mu\rho c^4\bar{k}}{\gamma}(3\bar{k}^2 - 1)(\bar{k}^2 - 1). \tag{21}$$

The capillary wave velocity c must here be replaced by $-U$ and $\bar{k}$ (which is related to A in (20)) is given by (6). Notice that since $\bar{k} \geqslant 1$, D increases rapidly with $\bar{k}$ and hence (see (7)) with the amplitude; also when linearized (21) gives the usual result for damping of capillary waves.

The only energy available to contribute to I is that produced by the surface tension effects on the basic flow. We shall assume from this point that this basic flow is in fact a gravity wave. The surface is then curved, and we have a 'surface tension pressure' $-\gamma\kappa$ acting on the surface and not taken into account in the solution for the gravity wave. If the gravity wave is travelling in the negative x direction with velocity c, if θ is the angle which the surface makes with the x axis, positive on the forward face, and $\mathbf{q}$ the velocity in the gravity wave, the basic flow U in (17) is the velocity in a frame moving with the gravity wave, i.e. a vector $\mathbf{U}$ parallel to the surface, where

$$\mathbf{U} = \mathbf{q} + c\mathbf{i}. \tag{22}$$

The component of $\mathbf{q}$ normal to the surface is given by

$$\mathbf{U}\cdot\mathbf{n} = 0 = \mathbf{q}\cdot\mathbf{n} + c\mathbf{i}\cdot\mathbf{n}, \tag{23}$$

or

$$\mathbf{q}\cdot\mathbf{n} = c\sin\theta, \tag{24}$$

$\mathbf{n}$ being the outward unit normal. Also if in the frame moving with the gravity wave the surface is $\psi = 0$ where ψ is the stream function, increasing upwards,

$$\kappa = \frac{\partial U}{\partial \psi} \tag{25}$$

(Longuet-Higgins 1963, (2.3)). Thus the rate of working of the surface tension pressure is

$$\gamma \frac{\partial U}{\partial \psi} c \sin\theta \tag{26}$$

per unit length of surface. Since any fluctuations of pressure give rise to waves we take I to be the modulus of (26). Finally the x in (17) becomes a curvilinear co-ordinate s along the surface of the gravity wave and we have as energy equation

$$\frac{-\rho U^3}{\sqrt{(\gamma^2 - C\rho U^2)}}\frac{dC}{ds} = \left|\gamma \frac{\partial U}{\partial \psi} c\sin\theta\right| - \frac{4\mu\rho U^4 \bar{k}}{\gamma}(3\bar{k}^2-1)(\bar{k}^2-1). \tag{27}$$

It might at first sight seem rather odd to use in I a surface tension pressure from a surface which, since it is covered with capillary waves, does not exist. However with s as a curvilinear co-ordinate the mean pressure over one capillary wave will vary with s as there is a curvature κ superimposed on the curvature already taken into account in the capillary wave theory. Thus as an approximation it seems to be in the spirit of the Whitham type of theory. Another point of difficulty is that the energy of the I term goes entirely into capillary waves. If some of it did generate a gravity wave component, this would, of course, modify the basic gravity wave but we could not deal with this in the present state of water wave theory. The ultimate justification of (27) must therefore depend on experiment.

4. Results

Once U and θ are known as functions of s, (27) is an ordinary differential equation which is easily solved numerically. For U and θ we use the same approximation as that used by Longuet-Higgins (1963), where its limitations are discussed. This is not very accurate, but at the time of writing is probably the best available. It gives

$$\left.\begin{aligned} U &= c(1+A^2-2A\cos m\phi)^{\frac{1}{6}}, \\ \tan 3\theta &= -A\sin m\phi/(1-A\cos m\phi), \\ \text{with} \quad m &= 3g/2c^3 \\ \text{and} \quad \frac{\partial U}{\partial \psi} &= U\frac{\partial \theta}{\partial \phi}. \end{aligned}\right\} \tag{28}$$

Here c is the phase-velocity of the wave, and the parameter A increases with the gravity wave amplitude, being one for the wave of greatest height with its pointed 120° crest. For a steep gravity wave A is therefore just < 1. The velocity potential ϕ is zero at a crest and $\pm\pi/m$ at the next troughs. A convenient variable for numerical work is $\overline{\phi} = m\phi$; the d/ds in (27) is changed to a $d/d\phi$ by the relation $U = d\phi/ds$ on $\psi = 0$.

The only remaining difficulty is the choice of an initial value for C. It seems reasonable as a starting point to take C zero in a gravity wave trough. If this is assumed and (27) is solved following the flow of energy (i.e. against the stream for capillary waves) C does not return to zero in the next trough, although it is very small. If however the integration is carried on over a second wave the value in the next, third, trough is the same as that in the second, i.e. a periodic solution is found. Actually the important results are not noticeably different between the two waves, so although some are presented for the periodic case, most have been calculated simply with C zero in the first trough.

Other things which can be calculated at the same time are the number of capillary waves, n, in a distance s, from

$$\frac{k}{2\pi} = \frac{dn}{ds} = U\frac{dn}{d\phi}, \tag{29}$$

and the viscous dissipation in the gravity wave, calculated from the formula (18) already used for the capillaries.

Results were calculated using a standard library programme for the Leeds University KDF 9 computer and are presented in figures 1 to 5. In all the calculations the values $\mu = 0{\cdot}0178$, $g = 981$, $\gamma = 74$ c.g.s. units are used. Figures 1 to 3 show the distribution of capillary wave amplitude along a gravity wave from one trough to the next (the gravity wave is travelling from left to right, opposite to the direction used in the theory), for three values of the gravity wave speed and four steepnesses of the gravity wave. The number of capillary waves is also indicated. The steepness is plotted in terms of a parameter λ, where

$$\lambda = \frac{2c^4\rho(1-A)^{\frac{4}{3}}}{3g\gamma}, \tag{30}$$

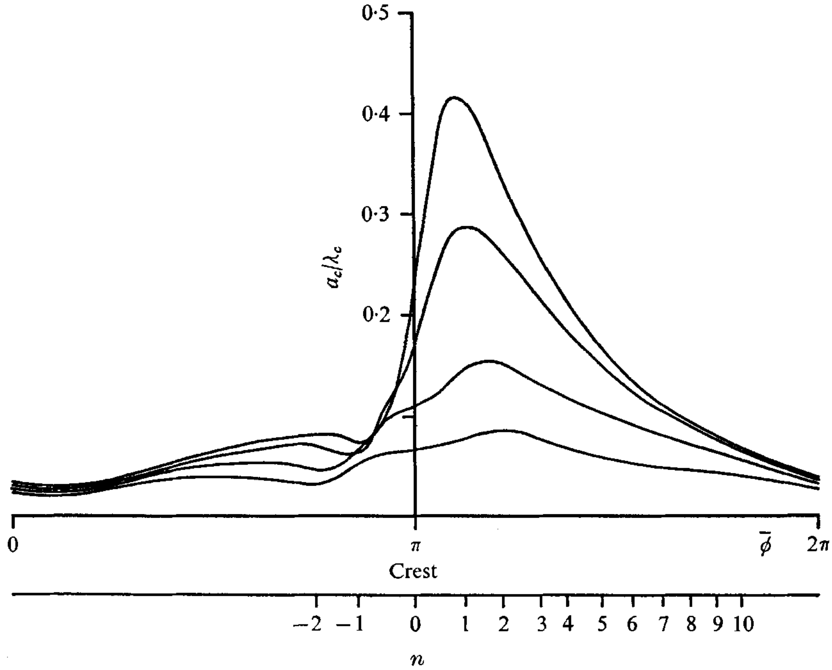

FIGURE 1. Capillary wave steepness distribution a_c/λ_c over one gravity wavelength for a gravity wave travelling from left to right with phase speed $c = 40$ cm/s and wavelength (on the approximation used) 6·83 cm. The second scale, n, gives the number of capillary waves measured from the gravity wave crest. From top to bottom the curves are for $\lambda = 1$ ($A = 0{\cdot}906$), $\lambda = 2$ ($A = 0{\cdot}842$), $\lambda = 5$ ($A = 0{\cdot}687$) and $\lambda = 9$ ($A = 0{\cdot}513$).

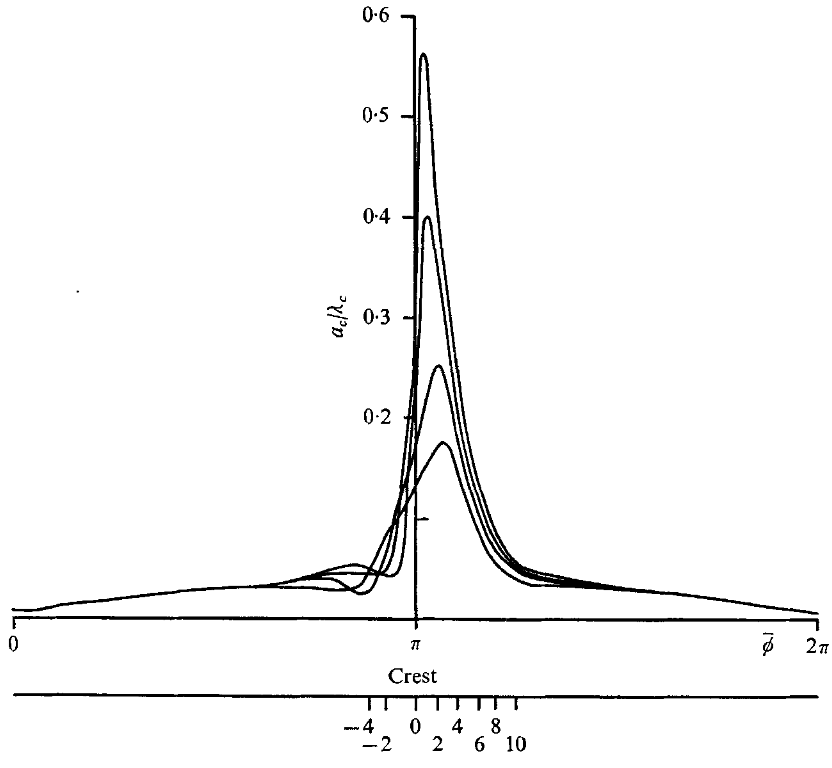

FIGURE 2. As figure 1 but for $c = 60$ cm/s and wavelength 15·37 cm. From top to bottom the curves are for $\lambda = 1$ ($A = 0{\cdot}972$), $\lambda = 2$ ($A = 0{\cdot}953$), $\lambda = 5$ ($A = 0{\cdot}907$) and $\lambda = 9$ ($A = 0{\cdot}856$).

with c and A as in equations (28) (λ must not be confused with wavelength). This parameter is used by Longuet-Higgins, and comes naturally into his linear solution, in which the capillary wave amplitude is simply a function of λ. The results here show that this is not the case for non-linear capillary waves but λ is used partly for comparison with earlier work and also because if A is sufficiently near to one, λ is simply related to the crest curvature κ_0 of the gravity wave (Longuet-Higgins 1963 (8.9)):

$$\lambda = g\rho/6\kappa_0^2\gamma. \tag{31}$$

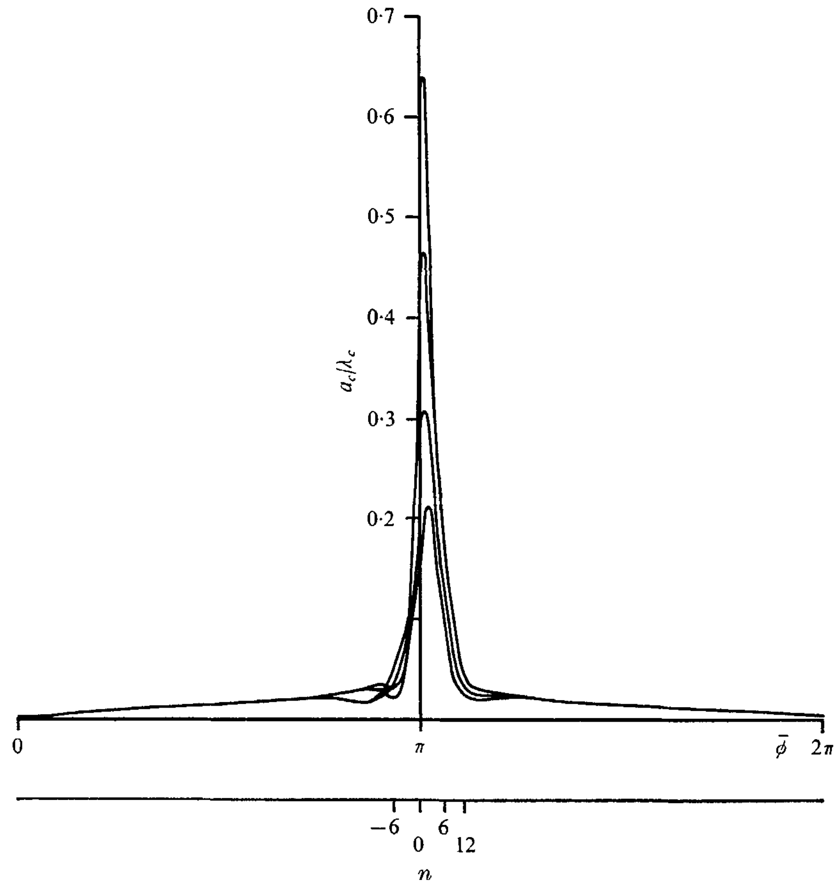

FIGURE 3. As figure 1 but for c = 80 cm/s and wavelength 27·33 cm. From top to bottom the curves are for λ = 1 (A = 0·988), λ = 2 (A = 0·980), λ = 5 (A = 0·961) and λ = 9 (A = 0·939).

The results are much as expected, showing if the gravity wave is sufficiently steep something of the order of 10 capillary waves of appreciable amplitude on the forward face of the gravity wave, with smaller amplitudes elsewhere, the amplitudes in the important region increasing with increasing gravity wave steepness. Comparison may be made with figure 3 of Longuet-Higgins' paper, but it should be noted that he uses a logarithmic scale for steepness and that his numerical values for steepness must be divided by π to compare with the present results. The general behaviour of the curves is seen to be similar, but Longuet-Higgins predicts rather lower amplitudes than the present results. He also makes some comparison with the experiments of Cox (1958), but the experimental work is for such a low wave speed (30·9 cm/s) and such a low value of A, of the order of $\frac{1}{2}$, that the gravity wave approximation used here is unlikely to be much use. As the present work depends very much on the actual shape of the gravity wave, it could not be expected to give a good comparison. The actual calculations for this case are given in figure 4 together with the experimental points, taken, not very accurately, from Longuet-Higgins' figure. Two calculations are presented, the upper one having the correct gravity wavelength, the lower the

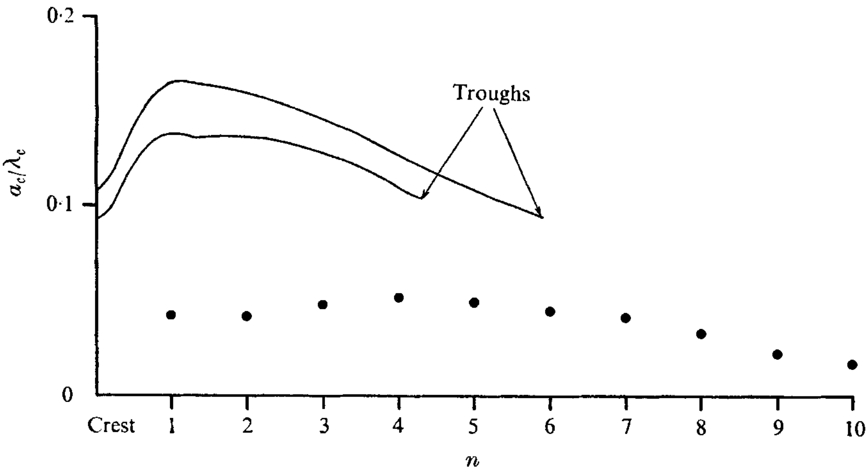

FIGURE 4. Comparison with experiment. The dots represent the experimental points for $c = 30·9$ cm/s and wavelength 4·7 cm taken from Longuet-Higgins (1963), based on the work of Cox (1958). The upper solid curve is the calculated capillary wave steepness for a gravity wave with $c = 33·17$ cm/s and wavelength 4·7 cm and the lower for $c = 30·9$ cm/s and wavelength 4·08 cm. In each case $\lambda = 2·92$, as estimated for the experiments, with $A = 0·633$ and 0·546 respectively.

correct gravity wave speed, as the two do not go together when we use the approximations in (28), both with $\lambda = 2·92$ as estimated by Longuet-Higgins. The amplitudes calculated appear much too large (whereas Longuet-Higgins' are too small) and there are not nearly enough capillary waves in the wavelength. This latter point is further evidence of the lack of accuracy of the assumed gravity wave form. Inspection of Cox's results as reproduced by Longuet-Higgins (figure 1) shows some capillary waves on the rear face of the gravity wave. This would be acceptable on the present theory, although not on Longuet-Higgins', as he chooses the position of a pole in his Fourier integral to ensure that all the waves are on the forward face.

All the curves show a rapid increase in a_c/λ_c and hence in the parameter C at the gravity wave crest. This means that in this region the slowly varying approximation breaks down and the numerical amplitude of the first large wave may be inaccurate. However, over the rest of the surface C does vary reasonably slowly. The stream velocity U is also varying most rapidly at the crest and slowly elsewhere and therefore imposes no further restriction.

Perhaps the most interesting results are those shown in figure 5. These curves give the ratio of the energy taken from the gravity wave by the capillaries and then damped out by viscosity, to the energy directly damped by viscous action on the gravity wave. In terms of the notation of § 3 the numerator of this ratio is the integral over the gravity wavelength of $D-I$, the total energy dissipated from the capillaries minus the total energy input to them, which always turns out

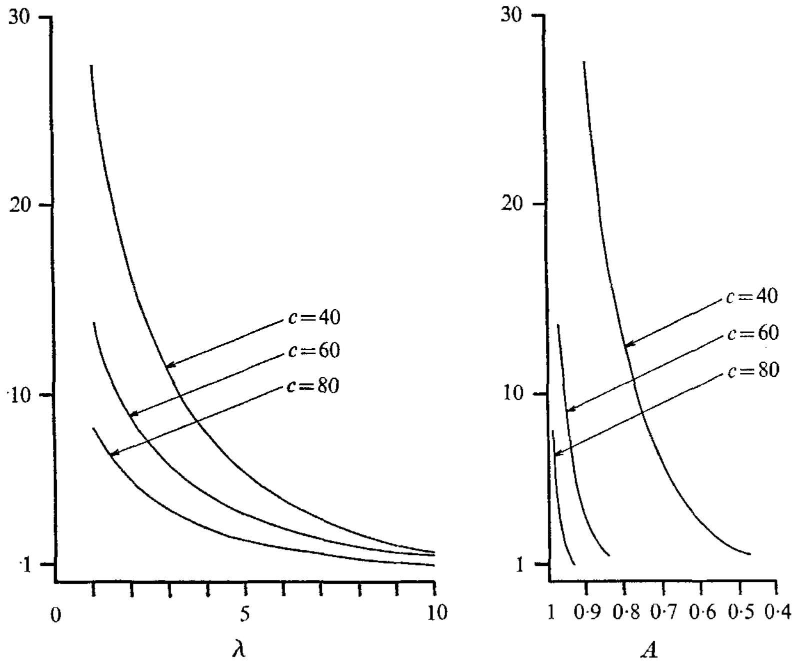

FIGURE 5. The ratio of energy removed from the gravity wave by the capillaries and then damped out by viscosity to energy damped out by direct viscous action on the gravity wave, shown both as a function of λ and as a function of A.

to be positive. As will be seen from figure 5 this ratio can be very large, especially as $A \to 1$, the value at which the gravity wave would break, so that conventional estimates of gravity wave damping are likely to be much too low for short steep waves, although adequate for long waves. Indeed the amount of energy removed from the shortest steep wave calculated (c = 40 cm/s, $\lambda = 1$, A = 0·906, wavelength 6·83 cm) is such as to almost damp it out completely in one wavelength, and the problem can hardly be regarded as steady. Another factor affecting the calculations for these short waves for c = 40 cm/s which can also be seen from figure 5 is that the values of A are not particularly near to 1, so that the gravity wave theory used here is not at all accurate. This means that c = 60 cm/s is probably the minimum speed at which any claim to accuracy can be made. Some experiments on the rate of decay of short, steep gravity waves would certainly be of interest; if the value of the surface tension coefficient could be varied the conclusions of figure 5 could be tested.

A final point concerns the maximum values of a_c/λ_c which appear in the figures. For $\lambda = 1$ these are 0·42 ($c = 40$ cm/s), 0·56 ($c = 60$ cm/s) and 0·64 ($c = 80$ cm/s). These are all lower than the theoretical maximum possible value of 0·73 (Crapper 1957), but are increasing with both c and λ, and will eventually exceed this figure. Actually for $c = 80$ cm/s, $\lambda = 2/3$ the maximum is $a_c/\lambda_c = 0{\cdot}76$. The maximum possible value is reached when two capillary waves bump into each other to enclose an air bubble in the trough; with damping present this will occur for some a_c/λ_c greater than 0·73, perhaps even as high as 1, since the biggest wave has to bump into a smaller one, but it seems clear that this will occur for some A very close to 1. Therefore, we suggest, the breaking of gravity waves is initiated by the breaking of these capillary waves. Since with surface tension taken into account the gravity wave cannot actually come to a point, this provides a more realistic physical mechanism for breaking.

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